

# Quasi-exact solvability of spiked harmonic oscillators

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## Abstract

Asymptotically quadratic potentials  $V_{(d)}(x) = d^2 - 2d|x| + x^2$  are considered as an elementary one-dimensional interaction model exhibiting a single-well shape at negative  $d = -\mu < 0$  and a double-well shape at positive  $d = \nu > 0$ . The existence of terminating, polynomial *alias* quasi-exact bound states is revealed and discussed. All of the  $N$ -plets of these states are constructed in closed form.

## Keywords:

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single well shape;  
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# 1 Introduction

Elementary one-dimensional potential

$$V(x) = V_{[d]}(x) = (|x| - d)^2 = \begin{cases} (x - d)^2 = x^2 - 2dx + d^2, & x > 0, \\ (x + d)^2 = x^2 + 2dx + d^2, & x < 0 \end{cases} \quad (1)$$

represents one of the simplest possible examples of a short-ranged perturbation of the ubiquitous harmonic oscillator. Still, in Eq. (1) the non-analyticity of  $V(x)$  in the origin looks “suspicious”. For this reason, in spite of the relative boundedness of the perturbation, the model seems never recalled in the context of perturbation theory. It sounds almost like a paradox that in the textbooks on quantum mechanics the role of the most popular illustrative example of the perturbation of harmonic oscillator is almost invariably played by the (nicely analytic) quartic anharmonicity  $\sim x^4$  yielding the *vanishing* radius of convergence of the resulting Rayleigh-Schrödinger perturbation series [1].

In our recent comment on quartic anharmonicities [2] we pointed out that the not too rational (perhaps, purely emotionally motivated) insistence of the major part of the physics community on the strict analyticity of the one-dimensional phenomenological potentials  $V(x)$  did also cause problems in the monograph [3]. Indeed, the author of this very nice review of the so called quasi-exactly solvable (QES) Schrödinger equations (offering also an extensive list of further relevant references) did not imagine that besides the best known sextic-polynomial QES potential  $V(x)$ , the QES status can be also assigned to its quartic-polynomial predecessor, provided only that we admit its non-analyticity at  $x = 0$  (cf. [2] for details).

Recently we returned to the QES problem and imagined that even the quartic polynomials  $V(x) = A|x| + Bx^2 + C|x|^3 + x^4$  which were studied (and assigned the QES status) in the latter reference need not still represent the “first nontrivial” case. Our subsequent study of the problem resulted in a rather unexpected conclusion that the role of the simplest nontrivial QES quantum model can in fact be played by the spiked harmonic oscillator (1).

A concise constructive demonstration of this assertion is to be presented in what follows.

## 2 Quasi-exact solutions

### 2.1 Heuristics

A non-constructive proof of the QES property of our model is elementary. In the single-well scenario we choose the special value of  $d = d^{(QES)} = -1$  and inserted the nodeless function

$$\psi_0^{(QES)}(x) = \begin{cases} (1 + x) e^{-(x^2/2+x)}, & x > 0, \\ (1 - x) e^{-(x^2/2-x)}, & x < 0 \end{cases} \quad (2)$$

(which belongs to  $L^2(\mathbb{R})$ ) in Schrödinger equation

$$-\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x). \quad (3)$$

Clearly, function (2) obeys the standard matching conditions in the origin (i.e., it has a continuous logarithmic derivative there) and it solves our bound-state problem (1) + (3) at  $E = E_0^{(QES)} = 3$ . Thus, QES expression (2) represents the *exact* even-parity ground-state wave function. In Fig. 1 our QES choice of parameters has been used for illustration purposes. The similar elementary QES feature also occurs after one moves to the double-well dynamical regime.

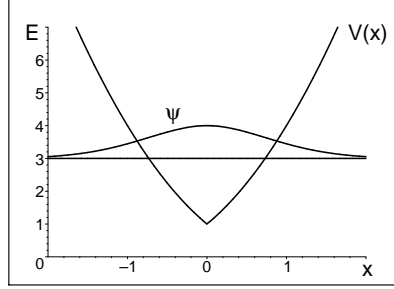


Figure 1: Ground-state energy and wave function in the single-well QES version of potential (1) at  $d = -\mu = -1$ .

Let us add that in the standard literature the QES property has only been assigned to the 1D potentials which were manifestly analytic along the whole real line [4, 5, 6]. The present, non-analytic generalization of the QES property has only been proposed, very recently, in Ref. [2] where the non-analytic QES construction has been preformed for the phenomenologically important class of quartic anharmonic oscillators. Our present real QES wave function (2) may be also recalled as providing a closed-form illustration of the fact that besides the potential, also our wave functions are non-analytic at  $x = 0$  since

$$\frac{d^3}{dx^3}\psi_0^{(QES)}(0^+) = 2 \neq \frac{d^3}{dx^3}\psi_0^{(QES)}(0^-) = -2. \quad (4)$$

We are persuaded that the descriptive merits of the similar generalized QES constructions overweight their not too essential feature of non-analyticity at  $x = 0$ . In this sense the existing literature would certainly deserve to be completed.

## 2.2 Systematic approach

After one inserts potential (1) in Schrödinger Eq. (3) it is possible to assume the existence of the even-parity and/or odd-parity QES bound-state solutions in the most general form of the standard normalizable polynomial ansatz such that, on the positive half-axis,

$$\psi^{(QES)}(x) = \psi(x) = e^{-x^2/2+dx} \times \sum_{k=0}^N a_k x^k, \quad x \geq 0, \quad a_N \neq 0. \quad (5)$$

We may accept the normalization convention

$$\begin{cases} \psi(0) = 1, & \psi'(0) = 0, \\ \psi(0) = 0, & \psi'(0) = 1, \end{cases} \quad \text{i.e.,} \quad \begin{cases} a_0 = 1, & a_1 = -d, \\ a_0 = 0, & a_1 = 1, & a_2 = -d, \end{cases} \quad \begin{matrix} \text{parity} = \text{even}, \\ \text{parity} = \text{odd}. \end{matrix} \quad (6)$$

With  $a_{N+1} = a_{N+2} = \dots = 0$  and after appropriate insertions and elementary algebra the QES solvability of the model appears equivalent to the validity of the set of linear recurrences

$$(E - 1 - 2n) a_n + 2d(n+1) a_{n+1} + (n+1)(n+2) a_{n+2} = 0, \quad n = 0, 1, \dots, N. \quad (7)$$

From the last item with  $n = N$  we get the QES value of energy  $E = 2N + 1$  so that we are left with an  $N$ -plet of relations between the  $N - 1$  unknown coefficients  $a_2, a_3, \dots, a_N$  and an unknown, QES-compatible value (or rather a multiplet of values) of the shift  $d = d^{(QES)}$ .

Starting from the choice of  $N = 0$  and  $E = 1$  we find that the odd solution cannot exist while the well known even-parity solution only exists at  $d = 0$ . In the next step with  $N = 1$  and  $E = 3$  the single constraint  $2a_0 + 2da_1 = 0$  only admits the pure  $d = 0$  harmonic oscillator in the odd-parity case while the first nontrivial QES solutions with  $d = \pm 1$  emerges in the even-parity scenario.

Once we skip the harmonic-oscillator solutions and demand that  $d \neq 0$  we may set  $N = 2$  and  $E = 5$  in Eq. (7). Then we have to satisfy the set of two relations

$$4a_0 + 2da_1 + 2a_2 = 0, \quad 2a_1 + 4da_2 = 0. \quad (8)$$

This leads to the two easy odd-parity solutions with  $d = d_{\pm} = \pm 1/\sqrt{2}$  and  $a_2 = -d$ . They become accompanied by the equally easy even-parity solutions with  $a_2 = 1/2$  and  $d = \pm\sqrt{5/2}$ .

Next we choose  $N = 3$  and find that the odd-parity construction degenerates to the triplet of relations

$$2 + 2da_2 + 3a_3 = 0, \quad d + a_2 = 0, \quad a_2 + 3da_3 = 0$$

with solution  $a_3 = 1/3$ ,  $d = \pm\sqrt{3/2}$  and  $a_2 = -d$ . In parallel, the even-parity conditions

$$-2d + 2da_2 + 3a_3 = 0, \quad 3 - d^2 + a_2 = 0, \quad a_2 + 3da_3 = 0$$

lead to the slightly less trivial quadruplet of eligible QES shifts

$$d_{\pm,\pm} = \pm \sqrt{\frac{9 \pm \sqrt{57}}{4}}. \quad (9)$$

Each one of them defines the related two coefficients

$$a_2 = \frac{2d^2}{2d^2 - 1}, \quad a_3 = \frac{2d}{6d^2 - 3}. \quad (10)$$

## 2.3 General case

At any  $N$  we may study the QES sets of coupled nonlinear algebraic equations using the computer-assisted elimination technique of Gröbner bases. In the even-parity case this algorithm generates certain polynomials  $P^{(N)}(d)$  which specify the QES-compatible shifts  $d$  as their zeros (see Table 1). The odd-parity QES solutions appear determined, similarly, via the values of  $a = a_2 = -d$  which coincide with the real zeros of other polynomials  $Q^{(N)}(a)$  (see Table 2).

Table 1: QES even-parity values of shifts  $d$  are defined, implicitly, as zeros of polynomials  $P^{(N)}(d)$ . A sample of the Gröbnerian elimination is added.

$N$	polynomial $P^{(N)}(d)$	elimination of $a_N$
2	$-5 + 2d^2$	$-1 + 2a_2 = 0$
3	$3 - 9d^2 + 2d^4$	$-8d + 2d^3 + 3a_3 = 0$
4	$27 - 28d^2 + 4d^4$	$11 - 2d^2 + 12a_4 = 0$
5	$-15 + 75d^2 - 40d^4 + 4d^6$	$29d - 19d^3 + 2d^5 + 15a_5 = 0$
$\vdots$	$\dots$	$\dots$

Table 2: QES values  $a = a_2 = -d$  determined as zeros of polynomials  $Q^{(N)}(a)$  (odd-parity case).

$N$	polynomial $Q^{(N)}(a)$	elimination of $a_N$
2	$-1 + 2a^2$	$-a + a_2 = 0$
3	$-3 + 2a^2$	$-1 + 3a_3 = 0$
4	$3 - 12a^2 + 4a^4$	$5a - 2a^3 + 6a_4 = 0$
5	$15 - 20a^2 + 4a^4$	$7 - 2a^2 + 30a_5 = 0$
$\vdots$	$\dots$	$\dots$

The most compact representation of the above set of results may be obtained when one imagines that the even- and/or odd-parity boundary condition (6) at  $x = 0$  may be reinterpreted as the respective initial, additional (i.e.,  $n = -1$ ) condition for recurrences (7). This enables us to solve these recurrences in terms of compact formulae.

**Theorem 1.** *In the even-parity QES case the general closed-form solution of recurrences (7) reads*

$$a_k = a_k^{(N)} = \frac{(-1)^k}{k!(k-1)!} \det \mathcal{P}^{(N,k)}(d), \quad k = (1, 2, \dots) 3, 4, \dots \quad (11)$$

*Its  $d$ -dependence is encoded in the tridiagonal  $k$  by  $k$  matrices*

$$\mathcal{P}^{(N,k)}(d) = \begin{pmatrix} d & 1 & 0 & 0 & \dots & 0 \\ 2N & 2d & 2 & 0 & \ddots & \vdots \\ 0 & 2N-2 & 4d & 6 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & 2(N-k+3) & 2(k-2)d & (k-1)(k-2) \\ 0 & \dots & 0 & 0 & 2(N-k+2) & 2(k-1)d \end{pmatrix}. \quad (12)$$

*These formulae must be complemented by the constraint  $a_{N+1}^{(N)} = 0$ , i.e., by the polynomial algebraic equation*

$$\det \mathcal{P}^{(N,N+1)}(d) = 0 \quad (13)$$

which represents an implicit definition of all of the admissible values of the specific, QES-compatible shift parameters  $d = d^{(QES)}$ .

*Proof.* is elementary and proceeds by mathematical induction. It is only necessary to keep in mind that in matrix (12) the first row reflecting the initial condition looks (and is) anomalous. That's why the form and validity of the  $k = 1$  solution  $a_1 = -d$  and also of the  $k = 2$  solution  $a_2 = d^2 - N$  are to be verified separately.  $\square$

**Theorem 2.** *In the odd-parity QES case the closed-form solution of recurrences (7) reads*

$$a_{k+1} = a_{k+1}^{(N)} = \frac{(-1)^k}{(k+1)!k!} \det \mathcal{Q}^{(N,k)}(d), \quad k = 1, 2, \dots \quad (14)$$

with the  $d$ -dependence encoded in the tridiagonal  $k$  by  $k$  matrices

$$\mathcal{Q}^{(N,k)}(d) = \begin{pmatrix} 2d & 2 & 0 & \dots & 0 \\ 2N-2 & 4d & 6 & \ddots & \vdots \\ 0 & 2N-4 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2(k-1)d & k(k-1) \\ 0 & \dots & 0 & 2(N-k+1) & 2kd \end{pmatrix}. \quad (15)$$

This must be complemented by the algebraic polynomial-equation definition  $a_{N+1}^{(N)} = 0$ , i.e.,

$$\det \mathcal{Q}^{(N,N)}(d) = 0 \quad (16)$$

yielding all of the admissible values of the specific, QES-compatible shift parameters  $d = d^{(QES)}$ .

*Proof.* Once one keeps in mind that in the odd-parity cases we have  $a_2 = -d$  at all  $N$  and  $k$ , the proof remains elementary and proceeds, along similar lines as above, by mathematical induction.  $\square$

### 3 Conclusions

In our present letter we sought for a nontrivial support of the existing tendencies [7] towards an extension of the class of “solvable” 1D potentials  $V(x)$  from completely analytic to non-analytic at some points (and, in particular, in the origin). We believe that these tendencies are natural and well motivated.

A strengthening and/or independent complement of our present arguments may be found in Ref. [8] where we sampled the methodical as well as practical gains of the approach by introducing another spiked, centrally symmetrized 1-D potential of the Morse-oscillator type. In spite of its non-analyticity in the origin the model was still shown to exhibit several features of the more conventional complete exact solvability.

For the generic 1-D interactions characterized by the various incomplete forms of solvability (like QES) the situation appears more complicated because the presence of the non-analyticities at some coordinates may restrict, severely, the practical as well as methodical applicability of the incomplete constructions in quantum physics. Still, one can feel encouraged by the existence of parallels between the one-dimensional and three dimensional central interactions. Indeed, the phenomenological appeal of the latter models is usually perceived as independent of the presence or absence of a non-analyticity in the origin. For illustration, *pars pro toto*, let us just mention the singular nature of the so called shape-invariant realizations of supersymmetry in quantum mechanics [9].

Summarizing, we believe that there are no reasons for an asymmetric treatment of solvability (and, in particular, of the QES models) in one and three dimensions. As long as in 3-D models the singularities in the origin are currently tolerated, a transfer of the *same* “freedom of non-analyticity” to 1-D systems is desirable. In our present letter we showed that such a change of paradigm is also productive (new models may be constructed) and satisfactory (in the QES setting, the classification of models may be formulated as starting from the most elementary quadratic polynomials at last).

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